ON THE DETERMINATION OF UNIFORMLY ACCURATE SOLUTIONS OF DIFFERENTIAL EQUATIONS BY THE METHOD OF PERTURBATION OF COORDINATES

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In finding approximate solutions of differential equations containing a certain parameter ε, a method is applied which consists in expanding the exact solution in a series of powers of ε and evaluating several coefficients. One proceeds in an analogous way when the small parameter enters not into the differential equation but only into the boundary conditions. Quite frequently the solution of zeroth order contains singular surfaces within the region of interest to us which are not part of the exact solution of the equation. In the solutions of higher order these singularities are not only preserved but grow even more pronounced. As a consequence the series in powers of ε becomes divergent in the vicinity of such singular surfaces and the method of a small parameter does not yield a solution. The indicated difficulties may be resolved if one expands in powers of ε not only the sought function but also the independent variables.

The new (perturbed) coordinates are selected from the condition of constructing a uniformly exact solution of the differential equation and are determined simultaneously with the solution to the problem [1]. Its cumbersomeness is one of the disadvantages of this method.

It is shown in this paper that the uniformly exact solution may be found not from the differential equation in terms of perturbed coordinates, but by a power series obtained as a result of the application of the usual method of a small parameter.

1. Ordinary differential equations. Let us consider a certain ordinary differential equation, containing a small parameter

$$L\left(\frac{du}{dx}, \frac{d^2u}{dx^2}, \ldots, \frac{d^ku}{dx^k}, u, x, \varepsilon\right) = 0 \tag{1.1}$$

We shall seek its solution in the form

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n(x) \tag{1.2}$$

Substituting (1.2) into the differential equation (1.1), we write it down also as a power series in ε. Equating to zero each coefficient of the series, we obtain a system of differential equations for the determination of $u_0(x)$, $u_1(x)$, ..., $u_n(x)$, ...

In the following we shall assume that all coefficients of the series (1.2), with the exception of perhaps $u_0(x)$, satisfy the linear equations. It is precisely under this condition that the method becomes effective. It permits to replace the nonlinear differential equation by a system of simpler equations.

The equations which are satisfied by $u_0(x)$, $u_1(x)$, ..., $u_n(x)$, ..., are written down in the form

$$L_0\left(\frac{du_0}{dx}, \frac{d^2u_0}{dx^2}, \ldots, \frac{d^ku_0}{dx^k}, u_0, x\right) = 0$$
$$L_1\left(\frac{du_n}{dx}, \frac{d^2u_n}{dx^2}, \ldots, \frac{d^ku_n}{dx^k}, u_n, x\right) = f_n \qquad (n = 1, 2, 3, \ldots) \tag{1.3}$$

Here f_n is a function which depends on x, u_0, u_1, ..., u_{n-1} and their derivatives. Thus the right-hand side of the equation for u_n is known, if the solutions of the previous equations of system (1.3) are found.

Let us assume now that in the region considered the solution of zeroth order has singular points in whose vicinity the series (1.2) ceases to exist. Lighthill [1] developed a method applicable to this case which permits to obtain expansions which possess uniform accuracy in the whole region. The essence of the method consists in that not only the dependent variable u is expanded into a series of ε, but also the independent variable x. Following [1], we introduce a new variable z by means of the formula

$$x = z + \sum_{m=1}^{\infty} \varepsilon^m x_m(z) = z + \delta \tag{1.4}$$

where $x_m(z)$ is a certain function of the variable z. We apply the method of a small parameter to Equation (1.1), written in terms of the new variable, i.e. we set

$$u = \sum_{n=0}^{\infty} \varepsilon^n U_n(z) \tag{1.5}$$

In place of the system of Equations (1.3) we obtain for the determination of the coefficient of the series (1.5) in terms of new coordinates

$$\begin{aligned} &L_0\left(\frac{dU_0}{dz}, \quad \frac{d^2U_0}{dz^2}, \ldots, \frac{d^kU_0}{dz^k}, U_0, z\right) = 0 \\ &L_1\left(\frac{dU_n}{dz}, \frac{d^2U_n}{dz^2}, \ldots, \frac{d^kU_n}{dz^k}, U_n, z\right) = F_n \end{aligned} \qquad (n = 1, 2, 3, \ldots) \tag{1.6}$$

where F_n depends on z, U_0, U_1, ..., U_{n-1} and their derivatives with respect to z. Beginning with $n = 1$, the differential equation for the determination of U_n will not be the same as that in terms of x. The right-hand sides of the equation will be different. The function F_n contains $x_1(z)$, $x_2(z)$, ..., $x_n(z)$ and their derivatives, i.e. the right-hand sides of the equations depend now on coefficients of the series (1.4). In Lighthill's method F_n is subjected to limitations in such a way, that the singularities do not grow stronger as the order of approximations increases. The limitations imposed on F_n lead in general to differential equations for the determination of $x_n(z)$. The system of Equations (1.6) in general is more cumbersome in terms of the variable z.

In order to avoid additional complications to the problem produced by the transformation (1.4), we attempt to establish a direct relationship between the solutions of the system of Equations (1.3) and (1.6). The functions u_n and U_n in expansions (1.2) and (1.5) are different functions of their arguments. Let us compare them. To do this, we replace x by Formula (1.4) and represent each coefficient of the series (1.2) as a series of powers of δ

$$u_n(x) = u_n(z + \delta) = \sum_{n=0}^{\infty} \frac{1}{k!} \delta^k \frac{d^k u_n(z)}{dz^k} \tag{1.7}$$

On the other hand

$$\delta^k = \left(\sum_{m=1}^{\infty} \varepsilon^m x_m(z)\right)^k = \varepsilon^k \sum_{m=0}^{\infty} c_m \varepsilon^m \tag{1.8}$$

where

$$c_0 = [x_1(z)]^k, \qquad c_m = \frac{1}{mx_1} \sum_{p=1}^{m} (pk - m + p)\, x_{p+1} c_{m-p}$$

Substituting (1.7) into the series (1.2) and taking into account

Equation (1.8), we obtain

$$u(x) = \sum_{n=0}^{\infty} \varepsilon^n \sum_{k=0}^{n} \sum_{m=0}^{n-k} c_{n-m-k} \frac{1}{k!} \frac{d^k u_m(z)}{dz^k} \tag{1.9}$$

$$c_0 = x_1^k, \qquad c_{n-m-k} = \frac{1}{(n-m-k)x_1} \sum_{p=1}^{n-m-k} (pk - n + m + k + p) x_{p+1} c_{n-m-k-p}$$

The function $u_m(z)$ in the expansion (1.9) also depends on z, just as $u_m(x)$ depends on x.

This permits to compare U_n and u_n for like arguments. For (1.5) and (1.9) it follows that

$$U_0(z) = u_0(z), \quad U_n(z) = u_n(z) + \sum_{k=1}^{n} \sum_{m=0}^{n-k} c_{n-m-k} \frac{1}{k!} \frac{d^k u_m(z)}{dz^k} \qquad (n \geqslant 1) \tag{1.10}$$

Formula (1.10) contains only the derivatives of the functions $u_0(z)$, $u_1(z)$, ..., $u_{n-1}(z)$, because $c_m = 0$ for $k = 0$, $m \geqslant 1$.

Using Formula (1.10), we can determine the solution of system (1.6), if $u_n(x)$ are known. On the other hand, with the aid of (1.10) we may evaluate $u_n(z)$ by $U_n(z)$.

For this purpose it is convenient to represent Formula (1.10) in the form

$$u_n(z) = U_n(z) - \sum_{k=1}^{n} \sum_{m=0}^{n-k} c_{n-m-k} \frac{1}{k!} \frac{du_m(z)}{dz^k} \tag{1.11}$$

Equation (1.11) is a recurrence formula to evaluate u_n by U_n. For $n - 1$ this same equation permits to express u_{n-1} by means of U_{n-1}, u_{n-2}, u_{n-3}, ..., u_0 and then eliminate u_{n-1} from the right-hand side of Formula (1.11). Carrying out this operation in succession, all terms under the summation sign will be represented in terms of functions U_{n-1}, U_{n-2}, ..., U_0 and coefficients c_m.

The new result may be written down in the form

$$u_n(z) = U_n(z) - \Phi_n(U_0, U_1, \ldots, U_{n-1}) \tag{1.12}$$

Formula (1.12) may be considered as that transformation of sought functions, which reduces the system of Equations (1.6) to the system of differential equations (1.3). Indeed, if instead of U_n a new sought function $U_n - \Phi_n(U_0, U_1, \ldots, U_{n-1})$ is introduced, then by virtue of (1.12) the difference $U_n - \Phi_n$ must satisfy the same differential equation as $u_n(z)$, i.e.

$$L_1\left(\frac{d(U_n-\Phi_n)}{dz}, \quad \frac{d^2(U_n-\Phi_n)}{dz^2}, \ldots, \frac{d^k(U_n-\Phi_n)}{dz^k}, \quad (U_n-\Phi_n), z\right) = f_n \tag{1.13}$$

where f_n is the same function of z, U_0, $U_1 - \Phi_1$, ..., U_{n-1}, as before of x, u_0, u_1, ..., u_{n-1}, respectively.

The invariance of the system of differential Equations (1.3) with respect to transformations (1.4) and (1.12) is proved above under the assumption of convergence of the series used in equation for deducing Formulas (1.10) and (1.11). But it is obvious that in the variable z the U_n may be reduced to the Formula (1.13) by means of a direct calculation of F_n and therefore, independently of whether or not the series (1.2) is convergent in the vicinity of some point.

Using the invariance of system (1.3) with respect to transformation (1.4) and (1.12), let us indicate a simpler procedure for the determination of the series (1.5) than given in [1].

Using the usual method of a small parameter, we find the solution of equation (1.1), i.e. we calculate the coefficients of series (1.1). Then we change the notation in these coefficients: instead of x in $u_n(x)$ we write z. The function $u_n(z)$ is a solution of Equation (1.13). With the aid of Formula (1.10) we find $U_n(z)$ by $u_n(z)$ which is a solution of the system of Equations (1.6). Then c_{n-m-k}, and as a consequence x_1, x_2, x_n, ... are selected such that $U_n(z)$ determined by Formula (1.11) contain singularities of order not higher than in U_0. Then the series (1.5) will be a uniformly accurate solution of Equation (1.1). In the region of convergence of the series (1.2) and (1.5), passing to the coordinate x by Formula (1.4), we obtain the original solution to Equation (1.1), i.e. that which was found by the usual method of a small parameter

If the boundary conditions are prescribed in that region where the series (1.2) is convergent, then a particular solution of the system of Equations (1.3) satisfying the prescribed boundary conditions may be taken as the original solution.

Then Formula (1.10) and transformation (1.4) permit to extend the solution of Equation (1.1) into the region where the series (1.2), found first, was divergent. Using first the series (1.2), we find next $U_n(z)$ for the same integral curve in the vicinity of a singular point.

If the boundary conditions are prescribed where the series (1.2) is divergent, then with the aid of Formula (1.10) and transformations (1.4), from the boundary conditions for the coefficients of the series (1.5), we may find the boundary conditions of the coefficients of the series (1.2), which again permits to use the series (1.2) in the analysis of the solution obtained.

In conclusion we note that the conditions necessary for the construction of a uniformly accurate solution are now imposed directly on the integrals of systems of differential equations (1.6), and not on the right-hand side of these equations. Formula (1.10) does not contain the derivatives of x_n with respect to z. The coefficients of the series (1.4) are found quite simply and their determination is no longer related to the necessity of solving differential equations.

2. Partial differential equations. By analogy to the case of ordinary differential equations, the process of constructing uniformly accurate solutions for partial differential equations may likewise be simplified. Let the function u, satisfying a partial differential equation, depend on n variables $x_1, x_2, \ldots, x_n$. The solution of the formulated problem depends on a small parameter ε, which may enter both the differential equations and boundary conditions.

The solution of the differential equation is sought in the form of a power series in ε

$$u = \sum_0^\infty \varepsilon^n u_n(x_1, x_2, \ldots, x_n) \tag{2.1}$$

We assume that in the region of variability of the variables $x_1, x_2, \ldots, x_n$ considered, the solution of zero order has singular surfaces in whose vicinity the convergence radius of the series (2.1) approaches zero. To construct a uniformly accurate solution, we introduce the transformation of coordinates

$$x_1 = z_1 + \sum_{n=1}^\infty \varepsilon^n f_n(z_1, x_2, \ldots, x_n) \tag{2.2}$$

and u is represented as a power series in ε with coefficients which depend on new coordinates $z_1, x_2, \ldots, x_n$

$$u = \sum_{n=0}^\infty \varepsilon^n U_n(z_1, x_2, \ldots, x_n) \tag{2.3}$$

Just as for a function of one variable, we can write a formula for a function of several variables which permits to determine $U_n(z_1, x_2, \ldots, x_n)$ by coefficients of the series (2.1). It is of the form

$$U_0(z_1, x_2, \ldots, x_n) = u_0(x_1, \ldots, x_n)\big|_{x_1=z_1} \tag{2.4}$$

$$U_n(z_1, x_2, \ldots, x_n) = u_n(x_1, \ldots, x_n)\big|_{x_1=z_1} + \sum_{k=1}^{n}\sum_{m=0}^{n-k} c_{n-m-k}\frac{1}{k!}\frac{\partial^k u_m(x_1, \ldots, x_n)}{\partial {x_1}^k}\bigg|_{x_1=z_1}$$

$$c_0 = {f_1}^k, \qquad c_{n-m-k} = \frac{1}{(n-m-k)f_1}\sum_{p=1}^{n-m-k}(pk+m+k-n+p)f_{p+1}\,c_{n-m-k-p}$$

The solution obtained by the usual method of a small parameter with the aid of Formula (2.4), may be extended into the region of singular surfaces of the zero order solution. This method of continuation of a solution is in no way different from the one presented in the previous section.

BIBLIOGRAPHY

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